Stable heteroclinic cycles for ensembles of chaotic oscillators

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We study the formation of synchronous clusters in ensembles of globally coupled chaotic oscillators. We reveal that at least three clusters of identical synchronization are formed in such a system for large enough values of coupling strength. Our main result is an unexpected intermittent process of clusterization. This process gives strong indication to the existence of a stable heteroclinic cycle.

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The dynamics of ensembles of chaotic oscillators is an object of active investigation and growing interest. Such systems arise in many fields, such as optics, electronics, chemistry, condensed matter, and biology (cf. [1]). Different types of synchronization have been found recently for ensembles of chaotic oscillators: phase [2], generalized [3], partial [4], and identical chaotic synchronization [5]. Recently, real systems which can be modeled by globally coupled oscillators have been reported in several fields (cf. [6]). For investigation of partial synchronization (clusterization) ensembles with global couplings are regarded as a very interesting particular case in which model investigations [7] as well as experimental observations [6] of this phenomenon have been carried out. We can distinguish effects correlated with the loss of synchronization as another group of phenomena: riddling of an attractor basin [8], bubbling [9], and on-off intermittency [10]. All these phenomena have also been found in laboratory experiments as well as in natural systems (cf. [1]). In this article we study another type of intermittency, which has not been related to synchronization before. Intermittency as observed experimentally in hydrodynamical problems [11] is explained by the existence of a stable heteroclinic cycle in the phase space of the corresponding model systems [12,13], which are usually ordinary differential equation systems for amplitudes of interacting modes. A heteroclinic cycle is a sequence of trajectories connecting a number of saddle invariant sets in a topological circle [13] (in many cases the invariant sets are fixed points). Robust heteroclinic cycles may exist in systems with symmetry. A trajectory approaching a heteroclinic cycle spends long periods near the saddle sets and makes fast transitions from one set to the next one. The intervals between these transitions increase exponentially without any limit, but the unavoidable presence of noise in experiments or a small violation of the symmetry limit the increasing of these intervals and lead to intermittency.

In this article we investigate synchronous cluster regimes in an ensemble of globally coupled oscillators. Using a recently proposed method of stability analysis [14], we find unexpected rich intermittent clusterization processes for this system, and we show that the main reason for this is that the regime approached by this process is unstable. We give an explanation of this phenomenon by demonstrating the presence of a stable heteroclinic cycle.

We analyze an ensemble of N globally coupled identical Rossler oscillators as a model system:

$$\dot{x}_i = -y_i - z_i, \quad i = \overline{1,N},$$
$$\dot{y}_i = x_i + Ay_i, \tag{1}$$
$$\dot{z}_i = B + z_i(x_i - C) + \frac{d}{N}\sum_{k=1,k\neq i}^{N} F(z_k).$$

A,B,C are parameters of each oscillator and d is the coupling strength. The coupling between elements is chosen to be a nondiffusion type, which is very widely distributed in nature [15,16]. In particular, we have chosen the nonlinear coupling function $F(z) = z/(1+\sigma z^2)$ which is used to describe the interaction between oscillators with a frequency control (lock) through the control signals from frequency locking loops [16]. This covers a wide range of physical systems, because this loop of frequency control has been built for different electronic/electrical oscillators. If an oscillator has this loop, then the equations for its frequency can be written and a coupling of our type can be organized. Examples of such physical systems are given in [16]. For our case the chosen function is used to simplify the dynamics of the system for increasing coupling strength, i.e., to regularize and then suppress oscillations. Several other coupling functions also give clusterization in this ensemble and we have chosen this one for simplicity in the numerical calculations. We fix the parameters so that each partial element is in a chaotic state ($A=0.2, B=0.2, C=5.7$). We also fix the parameter of the coupling function $\sigma=0.01$ and study the dependence of the ensemble dynamics on the coupling strength d.

It has been shown recently [14] that clusterization in a system of globally coupled maps can be caused by numerical pitfalls. We therefore use the numerical method suggested in [14] to avoid these pitfalls.

Suppose that elements of the ensemble can be divided into K groups, inside which the coordinates of the elements are close, i.e., $\|\mathbf{V_r}\| < 10^{\delta_1}$, where $\mathbf{V_r}: \{V_{xr} = x_i - x_r, V_{yr} = y_i - y_r, V_{zr} = z_i - z_r\}$ if the ith and the rth elements belong to the same group ($-9 \lesssim \delta_1 \lesssim -5$). Then, we describe the dynamics of the whole ensemble by a system of only K elements:

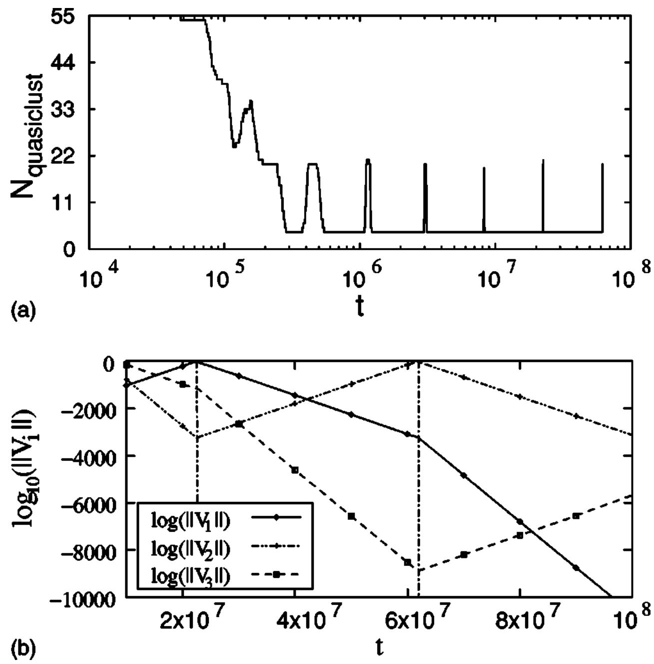

FIG. 1. Time series for the number of quasiclusters (a) and of norms for one difference from each cluster (b) in the case $N=55$, $d=5.2$ for system (1).

$$\dot{X}_i=-Y_i-Z_i, \quad i=\overline{1,K},$$

$$\dot{Y}_i=X_i+AY_i, \tag{2}$$

$$\dot{Z}_i=B+Z_i(X_i-C)+\frac{d}{N}\sum_{k=1}^{K}(m_k-\delta_{ik})F(Z_k).$$

Here, m_k is the number of elements in the kth group, δ_{ik} is the Kronecker symbol, i.e., $\delta_{ik}=0$ if $k\neq i$ and $\delta_{ii}=1$; $\Sigma_{k=1}^{K}m_k=N$. We next introduce linearized systems for the vectors of the difference variables $\mathbf{V_r}$:

$$\dot{V}_{xr}=-V_{yr}-V_{zr}, \quad r=\overline{K+1,N},$$

$$\dot{V}_{yr}=V_{xr}+AV_{yr}, \tag{3}$$

$$\dot{V}_{zr}=X_iV_{zr}+Z_iV_{xr}-CV_{zr}+\frac{d}{N}F'(Z_i)V_{zr}$$

in place of the remaining $N-K$ elements. We normalize these vectors periodically to avoid pitfalls due to finite precision and save (accumulate) the logarithms of the norms of these vectors $[\log_{10}(\|\mathbf{V_r}\|)]$. If any of these logarithms become larger than some value δ_2, we will restore the usual coordinates x_r, y_r, z_r. To summarize the method: when the coordinates of the elements become close with an increase in a dimensionless time (hereafter time), we switch from the initial system (1) to the system (2),(3). Note that K depends on time and means the number of temporal clusters, called quasiclusters.

We performed simulations for $N=55$, $d=5.2$ and chose $\delta_1=-8,\delta_2=-6$ [Fig. 1(a)]. Initial conditions were chosen randomly, so that none of the coordinates of the elements coincide within the precision 10^{δ_1} with any other, i.e., in the beginning each quasicluster (hereafter cluster) consists of only one element. After 50.000 units of time, the differences between the coordinates of 18 elements become sequentially less than 10^{δ_1}, i.e., the elements form one cluster of identical synchronization which is reached at about 100.000 units of time, and the number of quasiclusters decreases to 38. Then, in analogy with the previous case, two further clusters each consisting of 18 elements are formed and the number of quasiclusters decreases to four. The system spends about 100.000 units of time in this four-cluster regime. Then, one of the clusters scatters completely to separated elements and these form a cluster again (see explanations below). Such a scattering of a cluster repeats aperiodically. The time that the system spends in the unstable four-cluster regime between scattering of one of the clusters increases exponentially [see Fig. 1(a)]. Thus, we have obtained a complicated clusterization process.

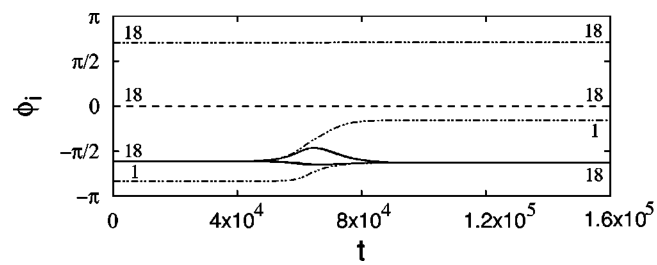

FIG. 2. Time series for phases of the elements of the system (1) in the interval of cluster scattering. Numbers of elements with equivalent phases for four-cluster regimes at the start and end points of this interval are shown. $N=55$, $d=5.2$

Let us consider the dependence of the logarithms of norms $\log_{10}(\|\mathbf{V_r}\|)$ on time [Fig. 1(b)]. One norm increases but in parallel the other two decrease. The increasing norm leads to the scattering and consequently to the formation of the corresponding cluster. After that, the norm corresponding to another cluster increases. Each time the increase of the norm starts with a smaller value. As a result, the time during which the system remains in the cluster regime increases.

We now study the process of scattering and the subsequent reconstruction of a cluster in detail. Each time this process takes about 2×10^5 units of time. It is important to emphasize that the time series of all elements in the considered cluster regime are periodic. We introduce a geometric phase for these oscillations, $\phi_i=\arctan(x_i/y_i)|_{x_1+Ay_1=0}, i=\overline{2,N}$ [2]. Figure 2 presents time series for the phases of all elements of the ensemble in the interval of cluster scattering. First, that cluster with the smallest phase deviation from the separated element (from a cluster of only one element) scatters. After the deviation of one of the cluster elements exceeds some value, the other elements form a new cluster. A separated element appears to be close to the next cluster now. Thus, the clusters lose their stability strictly sequentially.

Transverse Lyapunov exponents were introduced in [14] for testing the stability of cluster regimes in ensembles of maps. The transverse Lyapunov exponents for oscillators are defined as $\lambda_i^k=\langle\log_{10}\mu_i^k(t)\rangle$, where $\mu_i^k(t)$ are eigenvalues of

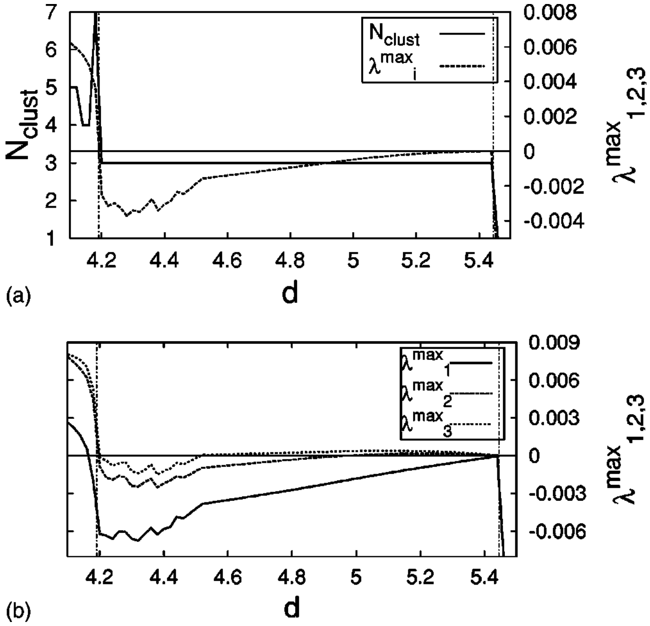

FIG. 3. (a) Dependence of the number of clusters and the maximal transverse Lyapunov exponents for a symmetric three-cluster regime on the coupling strength. (b) Maximal transverse Lyapunov exponents for a three-cluster asymmetric regime depending on the coupling strength. λ_1^{max}, λ_2^{max}, and λ_3^{max} obtained for clusters of 19, 20, and 21 elements, respectively ($N=60$).

the system (3), presented in evolution form: $\mathbf{V_r(t)} = D_i(t)\mathbf{V_r}(0)$. If all the transverse Lyapunov exponents are negative ($\lambda_i^{max}<0$, $i=\overline{1,K}$) we can, therefore, conclude that the cluster regime is stable. In the considered case, the values of the maximal transverse Lyapunov exponents for the four-cluster regime are $\lambda_1=0.000\,18$, $\lambda_2=-0.000\,45$, $\lambda_3=-0.000\,18$, which are in complete accordance with the dynamics of the norms. Thus, the cluster regime obtained in our numerical experiment is unstable.

This analysis allows us also to trace an analogy with processes observed in systems with stable heteroclinic cycles [13]. In the case studied here a stable heteroclinic cycle which connects saddle limit cycles is expected to exist, while for the majority of cases the heteroclinic cycle connects fixed points. For smaller values of the coupling strength ($4.74<d<4.88$) the intermittent clusterization process is obtained for the quasiperiodic cluster regime.

We analyze now conditions for obtaining this effect and describe the region where we can expect it. First we study an ensemble for which the number of elements is divisible by the number of clusters (three in our case); let us take $N=60$. Three clusters, each of which consists of 20 elements, are then obtained in the region $4.2\leqslant d\leqslant 5.44$ [Fig. 3(a)]. We call this regime symmetric, as a regime with an equal number of elements in the clusters. The maximal transverse Lyapunov exponents for each of the clusters ($\lambda_i^{max}, i=\overline{1,3}$) are negative in the whole region of d, i.e., the cluster regime is stable there [Fig. 3(a)]. Moreover, we have revealed that asymmetric regimes (regimes with a different number of elements in the clusters) are unstable in the major portion of this region [see Fig. 3(b)]. Hence, the symmetric regime is the only stable regime.

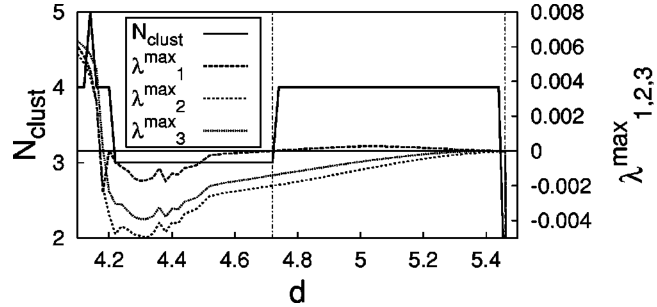

FIG. 4. Dependence of N_{clust} and the maximal transverse Lyapunov exponents for these cluster regimes on the coupling strength ($N=55$).

Now, we go back to $N=55$, where a three-cluster regime is realized in the region of smaller values of coupling strength and this regime is stable in the whole region (Fig. 4). Two clusters in this regime contain 18 elements, but the third one contains 19 elements. With increasing coupling strength, one element separates from the cluster with 19 elements, i.e., a fourth cluster, containing one element, is formed. The calculation of the maximal transverse Lyapunov exponents for each of these clusters, containing 18 elements, shows that the four-cluster regime is unstable in the whole region where it was observed (Fig. 4).

So, if the number of elements in the ensemble is divisible by the number of clusters, then it is possible to get a symmetric regime (equal number of elements in each cluster). Using the concept of transverse Lyapunov exponents, we have found that their stability regions are significantly wider than for nonsymmetric clusters. The stability regions of a K-cluster regime can be divided into two parts: with weak and strong convergence. In the region of strong convergence, the Lyapunov exponents of the symmetric K-cluster regime have large absolute values, and the nonsymmetric regimes close to the symmetric ones are stable. In the region of weak convergence, absolute values of the Lyapunov exponents of the symmetric regime are small, and even nearby nonsymmetric regimes are unstable.

If the number of elements in the ensemble is not divisible by the number of clusters, then there are only nonsymmetric regimes. We have found a stable regime in the region of strong convergence of the K-cluster regime. In the region of weak convergence, there is no stable regime. An intermittent clusterization process, which is expected to be correlated with a stable heteroclinic cycle, appears in this region. This process provides exponential increasing intervals during which the system remains in the unstable cluster regime.

We have also obtained this effect for a simpler case, an ensemble of only seven elements, which is the smallest possible number of elements for obtaining this effect because six elements form three symmetrical clusters and the seventh element breaks this symmetry. The process of intermittent formation of a four-cluster regime for this ensemble is obtained in the case $d=4.5$. Here, the behavior obtained is an intermittent clusterization process approaching the chaotic cluster regime (Fig. 5).

In conclusion, we have studied the effects of synchronous clusterization in an ensemble of chaotic oscillators with glo-

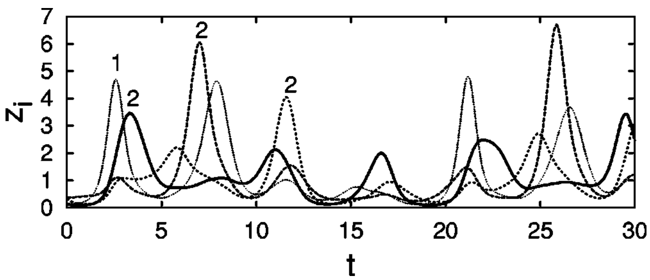

FIG. 5. Time series for all elements of the ensemble in the unstable four-cluster regime. $N=7$, $d=4.5$.

bal nonlinear couplings. We have revealed that three or more clusters of identical synchronization are realized in such a system for large enough coupling strength. The cluster regimes can be chaotic, quasiperiodic, or periodic depending on the coupling strength. Our main finding is an unexpected intermittent process of clusterization. The qualitative characteristics of this process are an exponential increase of the intervals during which the system remains in the cluster regime and a strict sequence of scatterings of clusters. We present the time series of norms as an explanation of the increasing intervals. The strict sequence of scatterings of clusters is qualitatively explained through the introduction of phases. These characteristics enable us to relate this process to a trajectory approaching a heteroclinic cycle. We trace a further analogy and show that the invariant sets corresponding to the cluster regimes are of saddle type, as they should be for the invariant sets connected in a heteroclinic cycle. Thus, we have carried out a detailed qualitative analysis of this phenomenon. We expect that these numerical findings will stimulate further theoretical research and experimental verification in various fields, such as ensembles of lasers, electrochemical oscillators, or neurons, where intervals of synchronous behavior of the system with intermittent periods of loss of synchronization should be observed. Such intermittent behavior is similar to the described clusterization process but without growing synchronous intervals. The mechanism of this intermittency is analogous to that obtained for hydrodynamical problems [13].

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